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Stable Model Reduction for Time-Domain Room Acoustics: A Structure-Preserving Formulation for Complex Boundaries

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ABSTRACT

This work presents novel structure-preserving formulations for stable model order reduction in the context of time-domain room acoustics simulations. A solution to address the instability in conventional model order reduction formulations based on the Linearized Euler Equations is derived and validated through numerical experiments. For the scenarios presented, the proposed formulation demonstrates 3.5 times speedup in calculating the full-order model compared to the unstable formulation. A critical stability analysis is performed to identify the eigenvalues of the reduced operator matrices falling on the right half of complex plane along with their out of phase eigen angles as the source of instabilities in the reduced system. The performance of the proposed MOR formulation is tested for perfectly rigid, frequency-independent, and locally reacting frequency-dependent boundary conditions in two-dimensional cases. The study shows that MOR using the stable formulation results in a 100-fold speedup. The proposed formulation is further evaluated to assess its impact on accuracy, computational speedup, and overall reduction potential of the system.

1 | Introduction

Room acoustics simulations are widely used for various applications in various industries, for example, architectural design [1, 2], sound system design [3], healthcare [4], hearing health research [5], entertainment [6], industrial facilities [7], hospitality [8, 9], transportation [10], virtual acoustics [11], and in the building industry [12]. These simulations are predominantly performed by geometrical acoustics (GA) methods [13, 14], such as ray tracing [15]. Although, GA methods are computationally efficient, they fail to capture the true wavelike nature of sound, which becomes particularly apparent

Abbreviations: FOM, full order model; POD, proper orthogonal decomposition; PSD, proper symplectic decomposition; ROM, reduced order model; TD, time domain.

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at low frequencies, where wave phenomena, such as diffraction and interference, play an important role. Consequently, simulation accuracy is known to suffer greatly due to degradation by the geometric approximation of treating sound propagation as rays using GA methods [16]. Another approach is to solve partial differential equations, that is, the wave equation in the time domain or Helmholtz in the frequency domain.

Various numerical methods have been applied to room acoustic simulations in the past, for example, the finite difference time domain (FDTD) [17, 18], the finite volume method (FVM) [19], the boundary element method (BEM) [20], the finite element method (FEM) [21, 22], the discontinuous Galerkin finite element method (DGM) [23, 24], and the particle-based method (PBM) [25]. Despite the superior accuracy of these wave-based methods, the major drawback of the high computational cost limits the usage to simulating wave propagation at low frequencies in small rooms. Even more challenging is the use of wave-based methods in interactive virtual acoustics, where, ideally, the simulation should occur within real-time constraints [26–28]. With the advancement of machine learning (ML) in recent years, deep learning (DL) based on physics-informed neural networks (PINN) is used to solve wave equations faster for massive seismic applications [29–31]. The approach used has employed high-order accurate numerical methods known as the spectral element method (SEM) [32], which can increase the cost efficiency of the simulations [33]. Another approach is to employ massively parallel computing architectures, such as graphics processing units (GPUs) [34–36]. Finally, domain decomposition methods [37] or wave/geometric hybrid models [14, 38–40] have also been shown to be useful to effectively accelerate computations by selectively applying techniques based on the resolution requirements of wave physics at frequencies of interest. Although the challenges mentioned above remain unsolved, the need for parametric studies to explore the design space to find the acoustically optimal design of the building space mandated the need to reduce the computation time. In the context of room acoustics, parametric studies were performed for cases with source location as a parameter using PINN and DL to achieve impressive computational speed after training [41, 42]. As the size and complexity of the problem increase, the training process experiences a proportionate rise in time requirements, posing challenges for efficient data handling.

A significant challenge in RBM techniques is maintaining stability during long-term time-domain (TD) simulations. Even when the full-order model (FOM) is inherently stable, instability issues can arise in the reduced-order formulation if not adequately addressed [43]. The stability of FOM can be inherited and maintained in the following approaches: (1) solving the problem in the frequency domain: Fourier domain (FD) and the Laplace domain (LD) where stability is ensured [44–46]. (2) Balanced truncation of the reduced basis (RB) to avoid 'mode-truncation instability' in the reduced-order model (ROM). (3) Optimization of reduced basis to preserve stability. (4) Structure-preserving formulation and reformulation of the original problem. (5) Stability ensuring projection of the reduced basis. (6) Use a structure-preserving timestepping scheme in the time domain.

FD approach of (1) is less appropriate for the problems of room acoustics with transient response, which is non-periodic in nature. In LD approach (1), the reconstruction of the time signal is a time consuming step in finding the optimal parameters using the Weeks method and is based on the impulse response of the TD computation [44]. The balanced truncation of (2) ensures stability but scales poorly with the size of the problem, resulting in an impractical method for acoustic simulations of large spaces [47]. In approach (3), the projection basis is modified a posteriori by solving a non-linear optimization problem to preserve stability [48–52]. The drawback of these approaches is that a trade-off between accuracy and stability occurs and, moreover, feasible solutions cannot be guaranteed in a finite number of iterations, restricting their use in practical problems. The stability ensuring formulation approach (4) can be considered as an a priori method for projection basis because the structure is preserved through the way the problem is formulated [53–55]. A previous study had successfully demonstrated stability through a structure-preserving formulation [56]. However, this approach relies on a problem-specific formulation of the governing equations of the vibroacoustics of the fluid structure. Moreover, this approach is restricted to frequency-independent (FI) boundary treatment. More recently, the same research group has extended structure-preserving model order reduction techniques to address frequency-dependent locally reactive admittance boundary conditions in time-domain vibro-acoustic simulations [57]. This further reinforces the growing interest in developing stable reduced-order models for complex boundary interactions. Another study demonstrated stable ROM using the structure-preserving Hamiltonian formulation for one-dimensional wave problems, but only at wave speeds below 1 m/s and without any dissipation at the boundaries [58]. There is no guarantee of stability with this approach for room acoustic applications, where sound dissipates at the actual speed of sound in air at the boundaries. (5) The stability of the original system can be preserved with symplectic structure in the formulation by careful symplectic projection [59]. Finally, (6) The structure preserving time marching (Stromer-Verlet) has shown preservation of stability in previous studies [58, 60, 61]. While approach (6) (Stromer-Verlet) is commonly used in combination with structure-preserving

formulations (approaches 4 and 5), stability can also be achieved using other schemes—such as generalized— α , provided the reduced system structure is preserved and sufficiently rich basis functions are used, as demonstrated in [56]. In this work, it is shown that the common numerical integrator selected as a Runge–Kutta method can also ensure stability provided that the combination of (4) and (5) is secured along with it.

In room acoustics, there are two additional factors that need to be considered for accurate modeling of wave propagation in realistic scenarios: (1) geometric flexibility, which allows simulation of wave interaction with curved and complex geometries; (2) high-order numerical accuracy to minimize wave dispersion and wave dissipation over long distances and to make the solver efficient. SEM and DGM offer geometric flexibility to model surface granularity in common-room shapes and high-order discretization with low dispersion and dissipation properties. In addition, the combination of RBM and SEM has been investigated in other areas, for example, for the Navier-Stokes equations [62, 63], where a potential synergy between high-order discretization and reduced basis methods is well demonstrated. After critical review of all aspects of MOR discussed earlier and the straightforward applicability of RBM to SEM, it has been decided to use the structure preserving formulation for the FOM solver using SEM [32] in the time domain. Stability ensuring symplectic basis projection in the MOR framework is chosen to demonstrate a stable ROM. We are not aware of any work on stable and structure-preserving MOR directly related to room acoustic simulations. However, there were some previous studies [58, 60, 64] in which MOR was applied to the wave equation in the time domain.

2 | Methods

2.1 | Governing Equations

The acoustic wave propagation in a lossless medium in a given physical and continuous domain Ω is described by the following second-order PDE,

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p + f(\mathbf{x}, t), \quad (1)$$

where $p(\mathbf{x}, t)$ is the acoustic pressure [Pa], $\mathbf{x}$ is the position vector in the domain Ω , c is speed of sound in air ($c = 343$ m/s), t is time [s] and $f(\mathbf{x}, t)$ is field source for sound propagation. ∇^2 is the Laplacian operator that corresponds to the dimension of the problem.

The initial conditions (ICs) are set by defining a Gaussian source pulse for the pressure part along with acoustic velocity as zero,

$$\begin{aligned} p(\mathbf{x}, t = 0, \mathbf{x}_0) &= \exp \left[-\left(\frac{\mathbf{x} - \mathbf{x}_0}{\sigma} \right)^2 \right], \\ \frac{\partial p(\mathbf{x}, t = 0, \mathbf{x}_0)}{\partial t} &= 0. \end{aligned} \quad (2)$$

where σ is the spatial variance of the source that determines the cutoff frequency to span and $\mathbf{x}_0$ is the position (m) of the source in the domain. The solution to the propagation of acoustic waves described by (1) can also be expressed as a system of coupled linear first-order PDEs described by the following formulations.

2.1.1 | Pressure-Velocity (p-v) Formulation

A set of linear first-order equations equivalent to (1) can be written based on the acoustic perturbation theory applied to Euler equations and commonly known as Linearized Euler Equations (LEE) [65]. They describe the wave propagation in a lossless medium with no mean flow and given by

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\rho c^2 \nabla \cdot \mathbf{v}, \\ \frac{\partial \mathbf{v}}{\partial t} &= -\frac{1}{\rho} \nabla p. \end{aligned} \quad (3)$$

where $\mathbf{v}(\mathbf{x}, t)$ is acoustic velocity (m/s) and ρ is the density of air ($\rho = 1.2$ kg/m³). The gradient operator ∇ is contextual, depending on the scalar and vector quantities of interest in the problem. To solve (3) numerically, the spatial derivatives

of the governing equations are discretized by introducing a globally continuous, time-independent test function $\phi(\mathbf{x})$, integrating over the spatial domain Ω . The resulting weak form is as follows

$$\begin{aligned} \int_{\Omega} p_t \phi \, d\Omega &= -\rho c^2 \left[\oint_{\Gamma} \phi \hat{\mathbf{n}} \cdot \mathbf{v} \, d\Gamma - \int_{\Omega} \mathbf{v} \cdot \nabla \phi \, d\Omega \right], \\ \int_{\Omega} \mathbf{v}_t \phi \, d\Omega &= -\frac{1}{\rho} \int_{\Omega} \nabla p \phi \, d\Omega. \end{aligned} \quad (4)$$

where $p_t = \frac{\partial p}{\partial t}$, $\mathbf{v}_t = \frac{\partial \mathbf{v}}{\partial t}$ and $\hat{\mathbf{n}}$ is the outward pointing normal vector of the spatial boundary Γ of domain Ω .

2.1.2 | Pressure-Potential (p- Φ) Formulation

The natural representation of the acoustic wave equation is in terms of velocity potential, Φ [66–68]. The system of equations in (3) can then be modified by introducing an acoustic velocity potential Φ such that

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z}. \quad (5)$$

equivalently, $\mathbf{v} = \nabla \Phi$. The system then can be written as

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\rho c^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right), \\ \frac{\partial \Phi}{\partial t} &= -\frac{1}{\rho} p. \end{aligned} \quad (6)$$

The weak formulation of the system (6) becomes

$$\begin{aligned} \int_{\Omega} p_t \phi \, d\Omega &= -\rho c^2 \left[\oint_{\Gamma} \phi \hat{\mathbf{n}} \cdot \nabla \Phi \, d\Gamma - \int_{\Omega} \nabla \Phi \cdot \nabla \phi \, d\Omega \right], \\ \int_{\Omega} \Phi_t \phi \, d\Omega &= -\frac{1}{\rho} \int_{\Omega} p \phi \, d\Omega, \end{aligned} \quad (7)$$

2.2 | Boundary Conditions

2.2.1 | Perfectly Reflecting Boundary Condition

In the case of *perfectly reflecting* (PR) boundaries, outgoing waves are entirely reflected back into the domain Ω , which is typical for rigid wall scenarios. Mathematically, this implies a zero surface normal velocity at the boundary: $\hat{\mathbf{v}}_n = \mathbf{v} \cdot \hat{\mathbf{n}} = 0$. As a result, the boundary term in the pressure equation vanishes.

2.2.2 | Frequency Independent Boundary Condition

Frequency independent (FI) boundaries assume constant absorption across frequencies and are characterized by a real-valued normal surface impedance Z [N-s/m⁴]. The corresponding normal velocity is given by:

$$\hat{\mathbf{v}}_n = \frac{p}{Z} = pY, \quad (8)$$

where Y is the admittance.

2.2.3 | Frequency Dependent Boundary Condition

The sound absorption in real world by many building materials is frequency dependent in nature. This type of absorption can be approximated by *locally reacting frequency dependent* (LR) boundary condition. The surface normal velocity which is frequency dependent in this case is given by

$$\hat{\mathbf{v}}_n(\omega) = \frac{\hat{p}(\omega)}{Z(\omega)} = \hat{p}(\omega)Y(\omega), \quad (9)$$

where ω is the angular frequency, $\hat{p}(\omega)$ and $\hat{\mathbf{v}}_n(\omega)$ are the Fourier transforms of the pressure and surface normal velocity. $Z(\omega)$ is the normal incidence surface impedance which is no longer a constant as it was in the FI case but is now a complex function of the angular frequency. $Y(\omega)$ is the surface admittance, for the sake of convenience, the surface admittance is used rather than the surface impedance. By applying the inverse Fourier transform to (9) yields the time-domain representation in the form of a convolution integral,

$$\hat{\mathbf{v}}_n(t) = \int_{-\infty}^t p(t')y(t-t') dt'. \quad (10)$$

To solve (10) accurately and efficiently, the method of auxiliary differential equations (ADEs) is used [32, 69, 70]. The boundary admittance is mapped to a multi-pole model, which is then inverse Fourier transformed and inserted into (10), yielding

$$\mathbf{v}_n(t) = Y_{\infty}p(t) + \sum_{k=1}^Q A_k \phi_k(t) + \sum_{k=1}^S 2 \left[B_k \psi_k^{(1)}(t) + C_k \psi_k^{(2)}(t) \right], \quad (11)$$

where Q and S are the number of real poles and complex conjugate pole pairs, respectively, used in the multi-pole approximation. The coefficients (Y_{∞} , A_k , B_k , C_k) correspond to parameters derived from the multi-pole mapping procedure, and the quantities (ϕ_k , $\psi_k^{(1)}$, $\psi_k^{(2)}$) represent internal state variables-referred to as accumulators-that evolve in time according to a system of ordinary differential equations.

$$\begin{aligned} \frac{d\phi_k}{dt} + \lambda_k \phi_k(t) &= p(t), \\ \frac{d\psi_k^{(1)}}{dt} + \alpha_k \psi_k^{(1)}(t) + \beta_k \psi_k^{(2)}(t) &= p(t), \\ \frac{d\psi_k^{(2)}}{dt} + \alpha_k \psi_k^{(2)}(t) - \beta_k \psi_k^{(1)}(t) &= 0. \end{aligned} \quad (12)$$

These additional equations must be solved at the boundary at each time step of the computation. (λ_k , α_k , β_k) are numerical coefficients of the multipole mapping. For the admittance model to be real and causal, λ_k and α_k must be ≥ 0 .

2.3 | Full Order Model

The FOM is often referred to as the high-fidelity solver in the RB literature. The FOM is obtained by discretizing the spatial derivatives using SEM and the temporal derivatives using a fourth-order explicit Runge–Kutta method. The SEM is a nodal variant of the finite element method, which combines the advantages of high-order accuracy and geometric flexibility [71]. For a more comprehensive understanding of the FOM solver used in this work, the readers are referred to Pind et al. [32], which provides a detailed description. The domain is divided into a set of non-overlapping elements Ω^k , $k = 1, \dots, N_{el}$ and a set of nodes is assigned to each element, making up a total of N_{fb} degrees of freedom (DOFs) on the mesh. The exact solution for the unknowns p and $\mathbf{v}$ is approximated by the full-order numerical solution denoted as $\mathbf{q}_{fb}(\mathbf{x}, t) = [p, u, v]^T$. This solution is represented as a series expansion of global basis functions $\mathcal{N}$, which are constructed by patching together Lagrange polynomial basis functions locally and weighted by time-dependent coefficients $a_i(t)$,

$$\mathbf{q}_{fb}(\mathbf{x}, t) = \sum_{i=1}^{N_{fb}} \mathcal{N}_i(\mathbf{x}) a_i(t). \quad (13)$$

A Galerkin formulation is obtained by selecting the test functions to be the same as the basis functions in the weak formulation (4). After discretization, the resulting semi-discrete system in 2D by the $p - \mathbf{v}$ formulation reads as

$$\begin{aligned}\mathcal{M}u_t &= -\frac{1}{\rho}S_x^k p, \\ \mathcal{M}v_t &= -\frac{1}{\rho}S_y^k p, \\ \mathcal{M}p_t &= \rho c^2(S_x^T u + S_y^T v - \mathbf{v}_n B),\end{aligned}\tag{14}$$

where (u, v) are the (x, y) components of acoustic velocity respectively, $\mathcal{M}$ is the mass matrix, S is the stiffness matrix and B is the matrix for surface treatment at the boundary. They are given by

$$\begin{aligned}\mathcal{M}_{ij} &\equiv \int_{\Omega} \mathcal{N}_j \mathcal{N}_i d\Omega \in \mathbb{R}^{N_{\text{fb}} \times N_{\text{fb}}}, \\ (S_m)_{ij} &\equiv \int_{\Omega} \frac{\partial \mathcal{N}_j}{\partial x_m} \mathcal{N}_i d\Omega \in \mathbb{R}^{N_{\text{fb}} \times N_{\text{fb}}}, \\ B_{ij} &\equiv \int_{\Gamma} \mathcal{N}_j \mathcal{N}_i d\Gamma \in \mathbb{R}^{N_{\text{fb}} \times N_{\text{fb}}},\end{aligned}\tag{15}$$

where m is the m th Cartesian coordinate. The global matrix operators, which are sparse, are determined by iterating over the elements and summing the local matrix operators resulting in diagonally dominant matrices. Thus, the inversion is trivial and it is the key feature of SEM for its spectral performance. The semi-discrete system in 2D (14) of the $p - \mathbf{v}$ formulation can be further written in *operator* form as

$$\begin{aligned}u_t &= \mathcal{L}_1 p, \\ v_t &= \mathcal{L}_2 p, \\ p_t &= \mathcal{L}_3 u + \mathcal{L}_4 v + \mathcal{L}_5 \mathbf{v}_n,\end{aligned}\tag{16}$$

where

$$\begin{aligned}\mathcal{L}_1 &= \mathcal{M}^{-1} \left(-\frac{1}{\rho} S_x^k \right), \\ \mathcal{L}_2 &= \mathcal{M}^{-1} \left(-\frac{1}{\rho} S_y^k \right), \\ \mathcal{L}_3 &= \mathcal{M}^{-1} (\rho c^2 S_x^T), \\ \mathcal{L}_4 &= \mathcal{M}^{-1} (\rho c^2 S_y^T), \\ \mathcal{L}_5 &= \mathcal{M}^{-1} (-\rho c^2 B).\end{aligned}\tag{17}$$

In the $p - \Phi$ formulation, the exact solution for unknowns p and Φ can be approximated by the full-order numerical solution, $\mathbf{q}_{\text{fb}}(\mathbf{x}, t) = [p, \Phi]^T$ to the system mentioned in (7) and applying the Galerkin principle, the semi-discrete form in 2D for this formulation is given as

$$\begin{aligned}\mathcal{M}\Phi_t &= -\frac{1}{\rho}\mathcal{M}p, \\ \mathcal{M}p_t &= \rho c^2(S_{xx}^T \Phi + S_{yy}^T \Phi - \mathbf{v}_n B).\end{aligned}\tag{18}$$

S is the stiffness matrix given by,

$$(S_{mm})_{ij} = \int_{\Omega} \frac{\partial \mathcal{N}_j}{\partial x_m} \frac{\partial \mathcal{N}_i}{\partial x_m} d\Omega \in \mathbb{R}^{N_{\text{fb}} \times N_{\text{fb}}}.\tag{19}$$

Similarly, the semi-discrete system in 2D (18) of the $p - \Phi$ formulation can be written in *operator* form as

$$\begin{aligned}p_t &= \mathcal{K}_1 \Phi + \mathcal{K}_3 \mathbf{v}_n \\ \Phi_t &= \mathcal{K}_2 p,\end{aligned}\tag{20}$$

where,

$$\begin{aligned}\mathcal{K}_1 &= \mathcal{M}^{-1}(\rho c^2 S_{xx}^T) + \mathcal{M}^{-1}(\rho c^2 S_{yy}^T), \\ \mathcal{K}_2 &= -\frac{1}{\rho}, \\ \mathcal{K}_3 &= \mathcal{M}^{-1}(-\rho c^2 B).\end{aligned}\tag{21}$$

The third term in the pressure equation of (16) and the second term in the pressure equation of (20) on the right hand side from two formulations respectively, account for the boundary contribution by the type of boundary condition. They modify according to the definition of $\mathbf{v}_n$ in Sections 2.2.1–2.2.3. The semi-discrete system in *operator* form is then integrated in time using an explicit Runge–Kutta time-stepping method.

When comparing two formulations, it can be observed that the $p - \Phi$ formulation is lighter and faster to solve. The components of acoustic velocity can easily be computed from the acoustic velocity potential Φ . The other advantage of $p - \Phi$ formulation is that the number of governing equations that need to be solved remains the same in the case of 3D with an extra term in the pressure equation for the extra dimension as compared to the 2D case.

2.3.1 | Energy Conservation

This section describes the conservation of energy in the system represented by two formulations under the implementation of the boundary conditions discussed earlier. A Hamiltonian system is a dynamical system that is completely described by the scalar function $H(\mathbf{q}, \mathbf{p}, t)$, which is called Hamiltonian [72]. The variable $\mathbf{q}$ is generally called the *momentum* and $\mathbf{p}$ is called the *potential*. The evolution equations of $H(\mathbf{q}, \mathbf{p}, t)$ in partitioned ODE form are given by

$$\begin{aligned}\mathbf{q}_t &= \nabla_{\mathbf{p}} H(\mathbf{q}, \mathbf{p}), \\ \mathbf{p}_t &= -\nabla_{\mathbf{q}} H(\mathbf{q}, \mathbf{p}).\end{aligned}\tag{22}$$

The system represented by the $p - \Phi$ formulation is a Hamiltonian system. Consequently, the system arising from the $p - \Phi$ formulation (6) is a *Hamiltonian* of Φ and p and comparing with (22), $H_{(i)}(\Phi, p, t)$ can be derived as [19]

$$H_{(i)}(\Phi, p, t) = \left(\frac{\rho}{2} |\nabla \Phi|^2 + \frac{p^2}{2\rho c^2} \right) d\mathbf{x}.\tag{23}$$

In spatially discrete form, $H_{(i)}(\Phi, p, t)$ can be written as,

$$H_{(i)}(\Phi, p, t) = \left(\frac{\rho}{2} |(S_x^T \Phi) \mathcal{M}(S_x \Phi) + (S_y^T \Phi) \mathcal{M}(S_y \Phi)| + \frac{p^T \mathcal{M} p}{2\rho c^2} \right).\tag{24}$$

The energy balance for the system by $H_{(i)}(\Phi, p, t)$ can be written as,

$$\frac{d}{dt} H_{(i)}(\Phi, p, t) = - \oint_{\Gamma} \rho \frac{\partial \Phi}{\partial t} \hat{\mathbf{n}} \cdot \nabla \Phi \, d\Gamma.\tag{25}$$

It can also be written as,

$$\frac{d}{dt} H_{(i)}(\Phi, p, t) = -\mathcal{B}_{(b)},\tag{26}$$

where $H_{(i)} = H_{(i)}(t) \geq 0$ is the total energy contained in the acoustic field interior of the domain, and where $\mathcal{B}_{(b)} = \mathcal{B}_{(b)}(t)$ is a term representing power outflow. In spatially discrete form, $\mathcal{B}_{(b)}(t)$ can be written as,

$$\mathcal{B}_{(b)}(t) = p_b \mathcal{M} |S_x^T \Phi| + p_b \mathcal{M} |S_y^T \Phi|,\tag{27}$$

where p_b is the pressure at the boundary. Similarly, in the case of the $p - \mathbf{v}$ formulation, the total energy by $E_{(i)}$ is given by

$$E_{(i)}(p, \mathbf{v}, t) = \int_{\Omega} \left(\frac{\rho}{2} |\mathbf{v}|^2 + \frac{p^2}{2\rho c^2} \right) d\mathbf{x}.\tag{28}$$

In spatially discrete form, $E_{(i)}(p, \mathbf{v}, t)$ can be written as,

$$E_{(i)}(p, \mathbf{v}, t) = \left(\frac{\rho}{2} \left| u^T \mathcal{M} u + v^T \mathcal{M} v \right| + \frac{p^T \mathcal{M} p}{2\rho c^2} \right). \quad (29)$$

The energy balance for the system, $E_{(i)}(p, \mathbf{v}, t)$ can be written as

$$\frac{d}{dt} E_{(i)}(p, \mathbf{v}, t) = - \oint_{\Gamma} p \hat{\mathbf{n}} \cdot \mathbf{v} d\Gamma. \quad (30)$$

For the system with PR boundaries, the right-hand terms in (25) and (30) vanish, resulting in the conserved total energy as a time-independent quantity.

The system with impedance boundary conditions (both frequency dependent and independent) is energy dissipative and, as a consequence, the *Hamiltonian* is no longer time independent. Such *Hamiltonian* is called a *Lyapunov* function, it requires that

$$B_{(b)} = Q_{(b)} + \frac{d}{dt} H_{(b)}(\Phi, p, t), \quad (31)$$

where $Q_{(b)} = Q_{(b)}(t) \geq 0$ represents dissipation, and where $H_{(b)} = H_{(b)}(t) \geq 0$ is the energy stored at the domain boundary. Then, the energy balance of the system can be written as

$$\frac{d}{dt} H(\Phi, p, t) = -Q_{(b)} \leq 0, \quad (32)$$

where the total energy $H = H_{(i)} + H_{(b)} \geq 0$ is non-increasing over time. Thus, a system with dissipation that is formulated to conserve energy would preserve the stability.

2.4 | Reduced Order Model

This section describes how the reduced basis method is used to reconstruct the full solution with much reduced spatial discretization operators of size N_{rb} , where $N_{rb} \ll N_{fb}$. The *solution manifold* is the set of all high-fidelity solutions to the parameterized problem under the variation of the parameter μ that belongs to the parameter space $\mathbb{P}$.

$$\mathbb{M}_{fb} = \{\mathbf{q}_{fb}(\mu) | \mu \in \mathbb{P}\} \subset \mathbb{V}_{fb}, \quad (33)$$

where $\mathbb{V}_{fb}$ is the discrete finite element approximation of the space. Note that when reducing time-dependent PDEs, the standard approach is to take time as being one of the parameters in the parameter space. The objective of the ROM is to approximate any member of this solution manifold with a small number of basis functions. The key assumption is that $\mathbb{M}_{fb}$ can be represented by the span of low number of appropriately chosen basis functions $\{\psi_i\}_{i=1}^{N_{rb}}$ with a small error in low dimension. The reduced basis space is defined as $\mathbb{V}_{rb} = \text{span}\{\psi_1, \dots, \psi_{N_{rb}}\}$. The reduced basis solution to the PDE as similar to (13) is as follows

$$\mathbf{q}_{fb}(\mathbf{x}, t, \mu) \approx \mathbf{q}_{rb}(\mathbf{x}, t, \mu) = \sum_{i=1}^{N_{rb}} \psi_i(\mathbf{x}) a_i(t, \mu), \quad (34)$$

and in matrix form,

$$\mathbf{q}_{rb}(\mathbf{x}, t, \mu) = \Psi a, \quad (35)$$

where $\Psi \in \mathbb{R}^{N_{fb} \times N_{rb}}$ is a matrix containing the reduced basis vectors as its column and $a \in \mathbb{R}^{N_{rb}}$ is a vector containing the time-dependent weighted coefficients for a given parameter μ .

The measure of how well $\mathbb{M}_{fb}$ can be approximated with a smaller subspace $\mathbb{V}_{rb}$ is given by the *Kolmogorov N-width* of $\mathbb{M}_{fb}$, defined as

$$d_N(\mathbb{M}_{fb}) = \inf_{\mathbb{V}_{rb}} \sup_{\mathbf{q}_{fb} \in \mathbb{M}_{fb}} \inf_{\mathbf{q}_{rb} \in \mathbb{V}_{rb}} \|\mathbf{q}_{fb} - \mathbf{q}_{rb}\|_{\mathbb{V}}. \quad (36)$$

The rate at which the reduced model approximates the full-order behavior, often measured by the Kolmogorov N -width, is inherently tied to the characteristics of the underlying partial differential equation and the associated initial and boundary conditions. While elliptic and parabolic problems typically exhibit an exponential decay in approximation error, hyperbolic problems tend to show a slower decay [73, 74], reflecting the greater difficulty in compressing their solution manifolds. In favorable cases, this decay can follow the trend $d_N(\mathbb{M}_{\text{fb}}) = \mathcal{O}(e^{\beta N_{\text{rb}}})$ which makes such problems particularly well suited for projection-based reduced-order modeling [44, 58, 60].

The method consists of two stages, a pre-computation (*offline*) stage and a query (*online*) stage. In the offline stage, an exploration of the parameter space is carried out to construct the low-dimensional problem-dependent basis that accurately captures the relevant underlying dynamic information of the original model. Thus, in this stage, the evaluation of the original FOM for multiple parameter values is required, resulting in a potentially costly procedure. The online stage consists of a projection onto a lower-dimensional space, which allows solving the parameterized PDE, at a reduced cost by virtue of a sufficiently small RB. It is crucial to construct the reduced basis at a minimal cost while still ensuring that the relevant physics is captured with a low-dimensional basis.

2.4.1 | The Offline Stage–Sampling and Basis Generation

Different methods have been proposed in the literature to construct the reduced basis optimally, such that the relevant underlying physics are captured in the ROM, while the cost of the offline stage is minimized. In this work, two approaches are considered, the proper orthogonal decomposition (POD), for the formulation $p - \mathbf{v}$ and the proper symplectic decomposition (PSD), for the formulation $p - \Phi$.

2.4.1.1 | Proper Orthogonal Decomposition. The POD is a mathematical technique used for data compression and low-dimensional approximation commonly used in the field of data analysis, signal processing, and fluid dynamics [75]. The technique provides an optimally ordered orthogonal basis in the least squares sense. ROMs are then obtained by truncating this optimal basis, retaining only the essential information that ensures the desired predefined accuracy is obtained. The generation of the basis consists of performing a sampling of the parameter space by discretization: $\mathbb{P}_{\Delta} \subset \mathbb{P}$ using a uniform spacing Δ . The resolution of Δ should be selected in such a way that the dynamics of the solution is accurately represented and $\mathbb{M}_{\text{fb}}(\mathbb{P}_{\Delta})$ is a good representation of $\mathbb{M}_{\text{fb}}$ with a resulting parameter set: $\text{span}\{\mu_1, \dots, \mu_{n_{\mu}}\}$. The full order solution $\mathbf{q}_{\text{fb}}(\mu_p)$ at a certain parameter μ_p , $p \in \mathbb{N}(n_{\mu})$ is called a *snapshot* and given as

$$\mathbf{q}_p = [\mathbf{q}^{(1)}(\mu_p) \dots \mathbf{q}^{(k)}(\mu_p) \dots \mathbf{q}^{(N_t)}(\mu_p)], \quad (37)$$

where $\mathbf{q}^{(k)}(\mu_p) \in \mathbb{R}^{N_{\text{fb}}}$ is a solution at a time step $k \in \mathbb{N}(N_t)$. Collecting all full-order solution matrices result in *snapshot matrix*. It is given by

$$\mathbf{S} = [\mathbf{q}_1 \dots \mathbf{q}_{n_{\mu}}] \in \mathbb{R}^{N_{\text{fb}} \times n_{\mu} N_t}. \quad (38)$$

The optimal spatial basis Ψ_S by POD can be obtained by solving the minimization problem as follows.

$$\Psi_S = \arg \min \|\mathbf{S} - \Psi \Psi^T \mathbf{S}\|_F^2 \ni \Psi^T \Psi = \mathbf{I}_{N_{\text{rb}}}, \quad (39)$$

where $\|\cdot\|_F$ denotes the Frobenius norm and the identity matrix $\mathbf{I}_{N_{\text{rb}}} \in \mathbb{R}^{N_{\text{fb}} \times N_{\text{fb}}}$ is with dimension $N_{\text{rb}} < n_{\mu} N_t$. The solution to the minimization problem can be achieved by singular value decomposition (SVD) on the snapshot matrix $\mathbf{S} = \mathbf{U} \Sigma \mathbf{V}^T$. The first N_{rb} columns of the left singular matrix $\mathbf{U}$ are the reduced basis $\Psi = [\mathbf{U}_1, \dots, \mathbf{U}_{N_{\text{rb}}}]$. The left singular matrix $\mathbf{U} \in \mathbb{R}^{N_{\text{fb}} \times l}$ and the right singular matrix $\mathbf{V} \in \mathbb{R}^{n_{\mu} N_t \times l}$ are orthogonal to each other and $\Sigma \in \mathbb{R}^{l \times l}$ is a diagonal matrix with singular values on its diagonal. Σ provides the measure of error of the reduced basis in the energy sense, where the dimension $l = \min(N_{\text{fb}}, n_{\mu} N_t)$. For the Galerkin projection, separate snapshot matrices, that is, S_p, S_u, S_v , must be constructed using a solution at all nodes in the time trajectory for each of the field quantities (p,u,v) using FOM by the formulation $p - \mathbf{v}$. The reduced bases Ψ_p, Ψ_u, Ψ_v are then constructed by performing a SVD on the corresponding snapshot matrices.

2.4.1.2 | Proper Symplectic Decomposition. A key property of Hamiltonian systems is their underlying symplectic geometry, which ensures conservation of the Hamiltonian function along the system's evolution on a symplectic manifold. This characteristic enables the use of symplectic model reduction methods that retain the structural integrity

of the full-order model (FOM) within the reduced-order model (ROM). It ensures that the stability properties of the FOM are inherited by the ROM. The system (22) can equivalently be written as

$$\mathbf{z}_t = \mathbb{J}_{2N_{\text{fb}}} \nabla_{\mathbf{z}} H(\mathbf{z}), \quad (40)$$

where $\mathbf{z} = (\mathbf{q}^T, \mathbf{p}^T)^T$ and

$$\mathbb{J}_{2N_{\text{fb}}} = \begin{bmatrix} 0_{N_{\text{fb}}} & I_{N_{\text{fb}}} \\ -I_{N_{\text{fb}}} & 0_{N_{\text{fb}}} \end{bmatrix}. \quad (41)$$

Suppose $\mathcal{A} \in \mathbb{R}^{2N_{\text{fb}} \times 2N_{\text{fb}}}$ represents a symplectic reduced basis that approximates the solution manifold $\mathbb{M}_H$ associated with equation (40). By applying the reduced-order approximation (34), we obtain

$$\mathbf{z} \approx \mathcal{A}a. \quad (42)$$

Substituting this approximation into equation (40) leads to

$$(\mathcal{A}a)_t = \mathbb{J}_{2N_{\text{fb}}} \nabla_{\mathbf{z}} H(\mathcal{A}a). \quad (43)$$

By left-multiplying both sides of equation (43) with the symplectic inverse of $\mathcal{A}$, denoted as $\mathcal{A}^+$, and applying the chain rule, we arrive at

$$a_t = \mathcal{A}^+ \mathbb{J}_{2N_{\text{fb}}} (\mathcal{A}^+)^T \nabla_{\mathbf{a}} H(\mathcal{A}a). \quad (44)$$

Given that $\mathcal{A}$ is symplectic, it follows that $(\mathcal{A}^+)^T$ is also symplectic, satisfying $\mathcal{A}^+ \mathbb{J}_{2N_{\text{fb}}} (\mathcal{A}^+)^T = \mathbb{J}_{2N_{\text{fb}}}$. Defining the reduced Hamiltonian $\tilde{H}(a) = H(\mathcal{A}a)$, the resulting reduced-order system is expressed as

$$a_t = \mathbb{J}_{2N_{\text{fb}}} \nabla_{\mathbf{a}} \tilde{H}(a). \quad (45)$$

This reduced formulation in equation (45) preserves the same structure as the original full-order system in equation (40), thereby retaining the symplectic structure and ensuring numerical stability of the ROM [58]. Different symplectic model reduction techniques exist in literature and in this work, *proper symplectic decomposition* (PSD) method, in particular the *cotangent lift* approach is employed [61]. It is an SVD-based method, and hence a similar minimization problem is solved as in the POD method. However, the symplectic bases are generally not orthogonal and therefore not norm bounded, making the minimization problem harder to solve than in the POD case. Suppose that the symplectic basis $\mathcal{A}$ is on the block-diagonal form

$$\mathcal{A} = \begin{bmatrix} \Psi_H & 0 \\ 0 & \Psi_H \end{bmatrix}, \quad (46)$$

where $\Psi_H \in \mathbb{R}^{N_{\text{fb}} \times N_{\text{fb}}}$ is a symplectic but orthonormal matrix, that is, $\mathcal{A}^T \mathbb{J}_{2N_{\text{fb}}} (\mathcal{A}) = \mathbb{J}_{2N_{\text{fb}}}$. The symplectic structure of $\mathcal{A}$ implies that Ψ_H should cover both the momentum and potential spaces of the Hamiltonian system. Therefore, a combined snapshot matrix must be defined and for the system based on the formulation $p - \Phi$ is given by

$$\mathbf{S}_c = [\mathbf{p}(\mathbf{x}, t_0) \dots \mathbf{p}(\mathbf{x}, t_{N_t}), \Phi(\mathbf{x}, t_0) \dots \Phi(\mathbf{x}, t_{N_t})]. \quad (47)$$

Then Ψ_H is given by selecting the N_{fb} leftmost singular vectors after performing an SVD on the combined snapshot matrix, $\mathbf{S}_c$, that is, $\Psi_H = [\mathbf{U}_1, \dots, \mathbf{U}_{N_{\text{fb}}}]$.

2.4.2 | The Online Stage

In the online stage, the Galerkin projection onto the span of the reduced basis results in reduction of the system's dimension and makes it faster to solve for new parameters. Assume a semi-discrete FOM representation for the system (16) in the assembled operator form

$$\mathbf{q}_t = \mathcal{L}q, \quad (48)$$

where $\mathcal{L}$ is the assembled operator which propagates the solution of the system with respect to time. The semi-discrete ROM is obtained by inserting the RB assumption (34) into (48), which yields

$$(\Psi a)_t = \mathcal{L} \Psi a, \quad (49)$$

that is,

$$a_t = \Psi^T \mathcal{L} \Psi a, \quad (50)$$

due to the orthogonality of the RB. The reduced spatial discretization operator of the ROM system, $\mathcal{L}_r = \Psi^T \mathcal{L} \Psi$, can be computed before the *online* stage. In the *online* stage, the semi-discrete system in (50) is integrated in time using standard time stepping methods for any parameter μ . The initial condition $q(\mathbf{x}, 0)$ is also projected, that is, $a(0) = \Psi^T q(\mathbf{x}, 0)$. Once the ROM has been solved for a , the output of interest is evaluated with $q = \Psi a$.

By inserting the RB assumption (34) into (16) along with $\mathbf{v}_n = 0$, the ROM for the system with PR boundaries becomes

$$\begin{aligned} (a_u)_t &= \Psi_u^T \mathcal{L}_1 \Psi_p a_p, \\ (a_v)_t &= \Psi_v^T \mathcal{L}_2 \Psi_p a_p, \\ (a_p)_t &= \Psi_p^T \mathcal{L}_3 \Psi_u a_u + \Psi_p^T \mathcal{L}_4 \Psi_v a_v. \end{aligned} \quad (51)$$

With $\mathbf{v}_n = \frac{p}{Z}$, the ROM for the system with FI boundaries becomes

$$\begin{aligned} (a_u)_t &= \Psi_u^T \mathcal{L}_1 \Psi_p a_p, \\ (a_v)_t &= \Psi_v^T \mathcal{L}_2 \Psi_p a_p, \\ (a_p)_t &= \Psi_p^T \mathcal{L}_3 \Psi_u a_u + \Psi_p^T \mathcal{L}_4 \Psi_v a_v + \Psi_p^T \mathcal{L}_5 \Psi_p a_p. \end{aligned} \quad (52)$$

For the system with LR boundaries, $\mathbf{v}_n$ is given by (11) at the boundary. As a result, a snapshot matrix for the normal velocity at the boundary $S_{\mathbf{v}_n}$ is needed to construct the corresponding reduced basis $\Psi_{\mathbf{v}_n}$. The ROM then is given by

$$\begin{aligned} (a_u)_t &= \Psi_u^T \mathcal{L}_1 \Psi_p a_p, \\ (a_v)_t &= \Psi_v^T \mathcal{L}_2 \Psi_p a_p, \\ (a_p)_t &= \Psi_p^T \mathcal{L}_3 \Psi_u a_u + \Psi_p^T \mathcal{L}_4 \Psi_v a_v + \Psi_p^T \mathcal{L}_5 \Psi_{\mathbf{v}_n} a_{\mathbf{v}_n}. \end{aligned} \quad (53)$$

To compute the boundary contribution $a_{\mathbf{v}_n}$ in the ROM, the reduced solution a_p is used to retrieve p at the boundary using $p = \Psi_p a_p$. This is then used when solving the ADEs in (12) to compute $\mathbf{v}_n$, which is then projected back to $a_{\mathbf{v}_n} = \Psi_{\mathbf{v}_n}^T \mathbf{v}_n$. This implies that the online stage of the ROM with LR BCs is not fully decoupled from N_{fb} .

Additionally, the ADEs system in (12) can equivalently be written as

$$\mathbf{a}_t = \mathbb{A} \mathbf{a} + \mathbb{B} p, \quad (54)$$

where $\mathbf{a} = \left(\phi_k^T, \psi_k^{(1)T}, \psi_k^{(2)T} \right)^T$ and

$$\mathbb{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_Q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_1 & -\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_1 & -\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_S & -\beta_S \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_S & -\alpha_S \end{bmatrix}. \quad (55)$$

$$\mathbb{B} = \begin{bmatrix} (1)_{\times Q} \\ 1 \\ 0 \end{bmatrix}_{\times S}. \quad (56)$$

Using the solution of system for accumulators, the snapshot matrix for accumulators is constructed and represented as,

$$\mathbf{S}_a = [\mathbf{a}_1 \dots \mathbf{a}_{N_t}] \in \mathbb{R}^{(Q+2S)N_b \times N_t}. \quad (57)$$

By performing SVD on $\mathbf{S}_a$, the reduced basis for the accumulators, Ψ_a is extracted as the leftmost singular vectors. Applying the reduced basis assumption (34) yields,

$$\mathbf{a} \approx \Psi_a \mathbf{a}_a. \quad (58)$$

Then by substituting this into (54), the additional reduced system for the frequency dependent boundary treatment can be written as,

$$(\mathbf{a}_a)_t = \Psi_a^T \mathbb{A} \Psi_a \mathbf{a}_a + \Psi_a^T \mathbb{B} \Psi_p \mathbf{a}_p. \quad (59)$$

Note that the ROM for the system with LR boundaries can be constructed without reducing the ADE system. The numerical experiments indicate that there are no noticeable improvements in computational efficiency with reduced ADE system but considerable improvements in achieving stable lower order solutions.

In the case of the $p - \Phi$ formulation, the symplectic projection ($\mathcal{K}_r = \Psi_H^T \mathcal{K} \Psi_H$) of the full-order solution (20) with PR boundaries onto the span of the symplectic reduced basis (46) results in

$$\begin{aligned} (\mathbf{a}_\Phi)_t &= \Psi_H^T \mathcal{K}_1 \Psi_H \mathbf{a}_p, \\ (\mathbf{a}_p)_t &= \Psi_H^T \mathcal{K}_2 \Psi_H \mathbf{a}_\Phi. \end{aligned} \quad (60)$$

With $\mathbf{v}_n = \frac{p}{Z}$, the ROM for the system with FI boundaries becomes

$$\begin{aligned} (\mathbf{a}_\Phi)_t &= \Psi_H^T \mathcal{K}_2 \Psi_H \mathbf{a}_p, \\ (\mathbf{a}_p)_t &= \Psi_H^T \mathcal{K}_1 \Psi_H \mathbf{a}_\Phi + \Psi_H^T \mathcal{K}_3 \Psi_H \mathbf{a}_p. \end{aligned} \quad (61)$$

For the system with LR boundaries, the main semi-discrete system is not completely decoupled from full order. Hence $\mathbf{a}_{\mathbf{v}_n}$ is computed in a way similar to the $p - \mathbf{v}$ formulation. The reduced for the main system is written as,

$$\begin{aligned} (\mathbf{a}_\Phi)_t &= \Psi_H^T \mathcal{K}_2 \Psi_H \mathbf{a}_p, \\ (\mathbf{a}_p)_t &= \Psi_H^T \mathcal{K}_1 \Psi_H \mathbf{a}_\Phi + \Psi_H^T \mathcal{K}_3 \Psi_{\mathbf{v}_n} \mathbf{a}_{\mathbf{v}_n}. \end{aligned} \quad (62)$$

The same additional reduced system as that of the $p - \mathbf{v}$ formulation is used to solve the accumulators to include frequency-dependent boundary conditions.

2.5 | Stability of the System

The performance of a room acoustic system, characterized by its impulse response, is inherently linked to the stability of its time-domain computations. Ensuring stability in both full-order and reduced-order systems is crucial, requiring investigation of mathematical conditions to develop accurate, structure-preserving MOR methods.

Both the full-order and reduced-order systems are *critically stable* if all generalized eigenvalues of their operator matrices in assembled form have a negative real part. Mathematically, the condition can be described as,

$$\Re(\Lambda(\mathcal{L}, \mathcal{L}_r, \mathcal{K}, \mathcal{K}_r)) \leq 0 \quad (63)$$

where Λ represents the eigenvalues of the systems under consideration as the array of complex numbers. The full system's operators in assembled form are,

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & \mathcal{L}_1 \\ 0 & 0 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 & \mathcal{L}_5 \end{bmatrix}, \quad (64)$$

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}_3 & \mathcal{K}_1 \\ \mathcal{K}_2 & 0 \end{bmatrix}. \quad (65)$$

A system with PR boundaries exemplifies an undamped system, as its energy is conserved and entirely contained. Ideally, for such an undamped system, the eigenvalues are purely imaginary, meaning that they lie along the central vertical axis of the complex plane, ensuring stability. This property is preserved in both the full-order and reduced-order systems designed using the $p - \Phi$ formulation. The formulation $p - \Phi$ conserves energy and maintains the positive semidefinite structure of the reduced operator matrix $\mathcal{K}_r$, which is critical to ensure stability.

The reduction of the operator matrix, $\mathcal{L}_r$, using the formulation $p - \mathbf{v}$ for systems with PR boundaries through the Galerkin projection results in the eigenvalues being shifted to the right half of the complex plane. This shift leads to instabilities that grow over time in reduced-order solutions.

Another crucial mathematical condition to ensure numerical stability in the time domain with time-stepping schemes is that all eigenvalues of the operator matrix, scaled by the size of the chosen time step k_d , must be within the stability region $\mathbb{S}$ offered by the chosen time-marching scheme. Mathematically,

$$k_d \Lambda(\mathcal{L}, \mathcal{L}_r, \mathcal{K}, \mathcal{K}_r) \in \mathbb{S} \quad (66)$$

Systems with both FI and LR boundary conditions serve as examples of damped systems and represent an initial boundary value problem (IBVP). In such systems, the presence of damping causes the eigenvalues of the operator matrices in the full-order system to shift into the left half of the complex plane, scattering along the negative real axis. For damped systems, satisfying the condition (66) is necessary but not sufficient to ensure stability.

An additional condition for preserving the structure in the reduced system is to ensure that the eigenvalues retain the same phase angle as those of the full-order system. However, for damped systems, the Galerkin and symplectic projections applied to the two formulations, respectively, alter the phase angles of the eigenvalues, causing a phase shift that deviates from the full-order system. Mathematically,

$$\Theta_\Lambda(\mathcal{L}_r, \mathcal{K}_r) \approx \Theta_\Lambda(\mathcal{L}, \mathcal{K}). \quad (67)$$

This issue arises from the inadequate construction of the reduced bases, which are generated using insufficient energy extracted from the full-order system. Both formulations discussed in the previous section rely solely on the energy within the system's interior to construct the reduced-order model. This approach overlooks the critical contribution of boundary energy, which is especially important in damped systems. As a result, the reduced bases do not fully capture the dynamics of the full-order system, leading to inaccuracies and potential instability in the reduced model.

However, as noted in (31), incorporating the boundary energy into the reduced basis construction allows the reduced system to inherit energy-based stability, thus preserving the structure of the full-order model.

2.6 | Stabilization by Structure Restoration

The formulation $p - \Phi$ is modified for damped systems to incorporate energy contributions at the boundaries by introducing an additional equation to solve and collect the boundary pressure at every time step. The resulting modified $p - \Phi$ formulation for the full-order system is expressed as follows:

$$\begin{aligned} p_{bt} &= \mathcal{K}_3 \mathbf{v}_n, \\ p_t &= \mathcal{K}_1 \Phi + \mathcal{K}_3 \mathbf{v}_n, \\ \Phi_t &= \mathcal{K}_2 p. \end{aligned} \quad (68)$$

Consequently, the combined snapshot matrix is adjusted to incorporate the solution for boundary pressure, resulting in the following modified formulation:

$$\mathbf{S}_{mc} = [\mathbf{p}(\mathbf{x}, t_0) \dots \mathbf{p}(\mathbf{x}, t_{N_t}), \Phi(\mathbf{x}, t_0) \dots \Phi(\mathbf{x}, t_{N_t}), \mathbf{p}_b(\mathbf{x}, t_0) \dots \mathbf{p}_b(\mathbf{x}, t_{N_t})]. \quad (69)$$

The reduced basis is then enriched with the energy of the full-order system. Specifically, Ψ_{mH} is constructed by selecting the N_{rb} singular vectors on the left obtained from the singular value decomposition (SVD) of the combined snapshot matrix $\mathbf{S}_{mc}$, that is, $\Psi_{mH} = [\mathbf{U}_1, \dots, \mathbf{U}_{N_{rb}}]$. The key advantage of this modified approach is that the reduced system can still be solved using the original, unmodified formulation while leveraging the modified reduced basis. This enhancement introduces energy-based stability to the reduced system.

3 | Numerical Experiments

3.1 | Pressure-Potential ($p - \Phi$) FOM Verification

In this section, the $p - \Phi$ FOM is first validated by comparing its results with both the analytical solution [76] and those obtained using the $p - \mathbf{v}$ formulation for the case with PR boundaries. Then a comparison is made for test cases involving impedance-type boundary conditions (BC) discussed in Section 2.2.

A rectangular domain of $4 \text{ m} \times 2 \text{ m}$ is used for the comparison as shown in Figure 1. First, the model is tested with the PR boundary condition. The model is initiated with a Gaussian pulse pressure distribution (Equation (2)) with $\sigma = 0.3 \text{ m}^2$. This yields a broadband initial pulse that has a -10 dB upper cutoff frequency of around 700 Hz . The pulse is centered at $\mathbf{x}_0 = (1.7, 1.2) \text{ m}$. The domain is meshed with an unstructured mesh of triangular elements represented by the polynomial order of the basis functions $P = 4$. The resolution of the mesh is chosen so that there are approximately 7 points per wavelength (PPW) at 700 Hz . This yields a mesh that has 178 elements (N_e) and $N_{fb} = 1509$ DOFs in total. A Courant-Friedrichs-Lewy (CFL) number of 0.2 is used, with a time step $\delta t = 1.5 \times 10^{-5} \text{ s}$, to ensure numerical stability and sufficient temporal resolution. The simulation time is 50 ms . The analytical solution from Reference [76] is expressed as an infinite series of integral terms; consistent with the original work, the summation was truncated after the first 120 terms, beyond which no discernible change in the impulse response was observed, indicating convergence. Figure 2a shows the impulse response at the receiver point, $\mathbf{r}_0 = (0.25, 0.22) \text{ m}$, with good comparison between the results using two formulations.

The L_∞ norm of relative error between impulse responses calculated by analytical solution and the formulation $p - \mathbf{v}$ is defined as $\epsilon_{ana-pv} = \|(p_a - p_{p-v})/p_a\|_\infty = 0.0027$. The L_∞ norm of relative error between impulse responses calculated by analytical solution and the formulation $p - \Phi$ is defined as $\epsilon_{ana-pphi} = \|(p_a - p_{p-\Phi})/p_a\|_\infty = 0.0026$. Similarly, the L_∞ norm of relative error between two impulse responses is defined as $\epsilon_{form} = \|(p_{p-v} - p_{p-\Phi})/p_{p-v}\|_\infty = 1.04 \times 10^{-4}$.

Secondly, the case is tested with FI boundaries using two surface impedances of $Z = [60016, 000] \text{ N-s/m}^4$. The corresponding values of the normal incidence absorption coefficient, α_0 , are $[0.9653 \ 0.0978]$, where $\alpha_0 = 1 - \left| \frac{Z - \rho c}{Z + \rho c} \right|^2$. Figure 2b shows the impulse response at the receiver point that is in good agreement with the results of two formulations. For the case with $Z = 16,000 \text{ N-s/m}^4$, the L_∞ norm of relative error between two impulse responses is $\epsilon_{form} = \|(p_{p-v} - p_{p-\Phi})/p_{p-v}\|_\infty = 8.6 \times 10^{-5}$.

Finally, the LR boundary condition is applied by modeling porous material mounted on a rigid backing at the boundaries. The thickness used is $d_{mat} = [0.05 \ 0.2] \text{ m}$ and the flow resistivity of the material is $\sigma_{mat} = 10,000 \text{ N-s/m}^4$. The characteristic impedance and the wave propagation constant of the porous material are obtained by using the Miki's model [77], and the surface impedance is computed using a transfer matrix method [78]. In multipole mapping, the material is mapped by vector fitting [79], with 4 poles. Figure 2c shows the good match between the results of two formulations for the frequency-dependent boundary condition. For the case with $\sigma_{mat} = 10,000 \text{ N-s/m}^4$, $d_{mat} = 0.2 \text{ m}$, the L_∞ norm of relative error between two impulse responses is $\epsilon_{form} = \|(p_{p-v} - p_{p-\Phi})/p_{p-v}\|_\infty = 6.6 \times 10^{-6}$.

Calculations using the $p - \Phi$ formulation are approximately 3.5 times faster than those using the $p - \mathbf{v}$ formulation under identical computational conditions. To illustrate, three cases are examined: (a) PR boundaries, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{mat} = 10,000 \text{ N-s/m}^4$, $d_{mat} = 0.05 \text{ m}$.

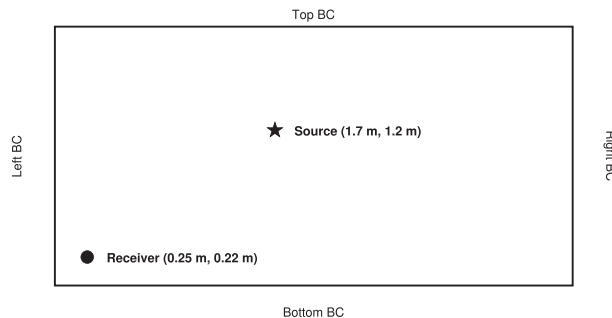


FIGURE 1 | 2D simulation domain ($4 \text{ m} \times 2 \text{ m}$) showing boundary conditions (BCs) applied on all edges for three cases: (a) PR boundaries, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{mat} = 10,000 \text{ N-s/m}^4$, $d_{mat} = 0.05 \text{ m}$. Source and receiver locations are also indicated.

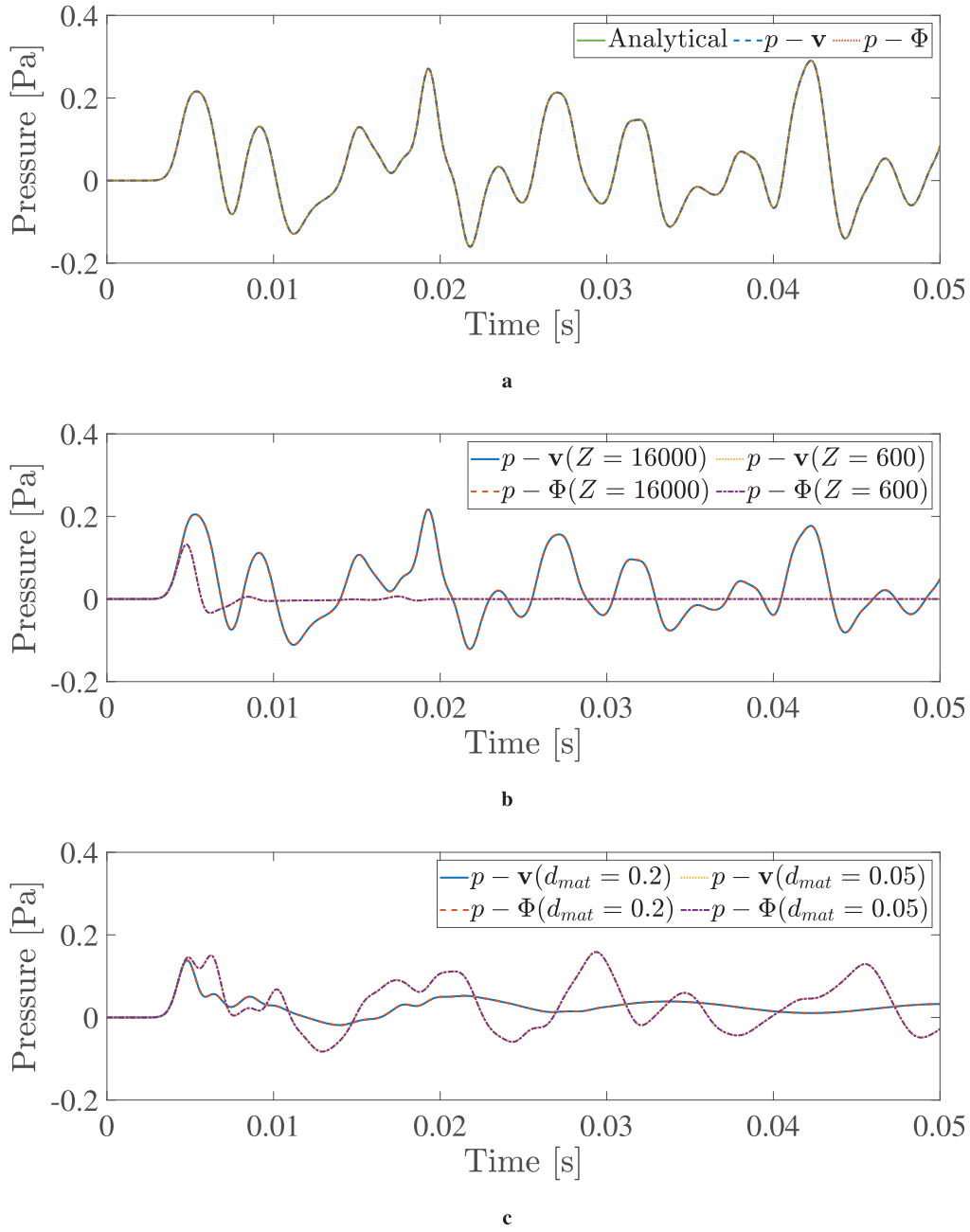


FIGURE 2 | Impulse responses calculated using the $p - \mathbf{v}$ and $p - \Phi$ formulations are shown for the following setup: source location $\mathbf{x}_0 = (1.7, 1.2)$ m, receiver location $\mathbf{r}_0 = (0.25, 0.22)$ m. Simulation parameters are: $P = 4$, $\delta t = 1.5 \times 10^{-5}$ s, $PPW = 7$, $N_e = 178$, $N_{fb} = 1509$ DOFs. (a) Perfectly reflecting boundary condition. (b) Frequency-independent boundaries. (c) Locally reacting frequency dependent boundary condition.

$Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{mat} = 10,000 \text{ N-s/m}^4$, $d_{mat} = 0.05$ m. Figure 3 presents the computational cost in CPU time for simulating 50 ms and highlights the speed-up factor ($CPU_{p-\mathbf{v}}/CPU_{p-\Phi}$), demonstrating the performance of the $p - \Phi$ formulation.

3.1.1 | Energy Conservation

Energy conservation for systems with boundary conditions PR, FI, and LR, as described in Section 2.3.1, is illustrated in Figure 4 on a normalized scale of E/E_0 , where E_0 represents the initial energy of the system. The energy $E_{(i)}(p, \mathbf{v}, t)$ calculated by the formulation $p - \mathbf{v}$ is normalized and compared to $H_{(i)}(\Phi, p, t)$ normalized using the formulation $p - \Phi$

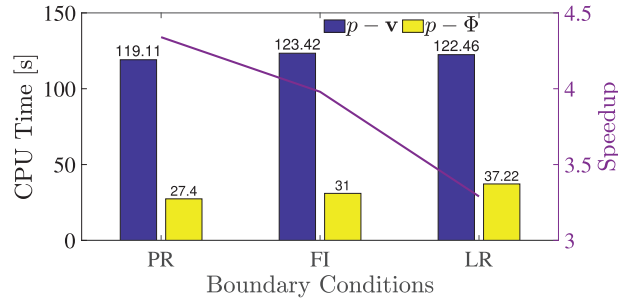


FIGURE 3 | CPU time and Speed-up comparison among $p - \mathbf{v}$ and $p - \Phi$ formulations for the cases: (a) PR boundaries, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05 \text{ m}$.

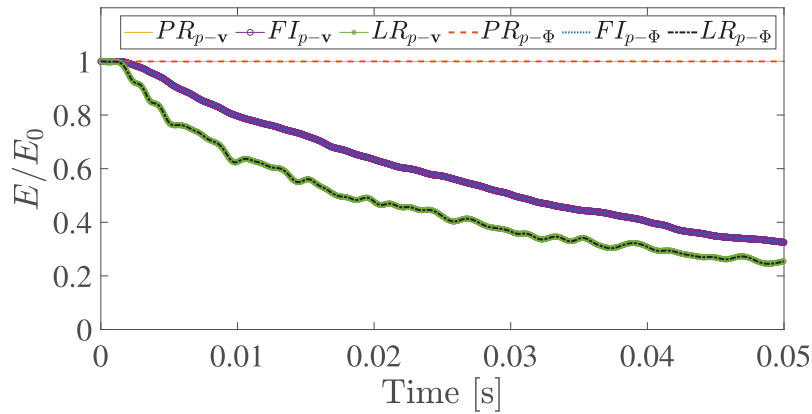


FIGURE 4 | Energy conservation. (a) PR boundaries, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05 \text{ m}$.

for three BCs. For PR boundaries, the energy remains constant over time as there is no dissipation at the boundaries. For the system with FI boundaries ($Z = 16,000 \text{ N-s/m}^4$), the energy is no longer constant, but varies over time due to the dissipation of the energy of the incident sound wave by the imposed frequency-independent impedance at the boundaries. Similarly, for the system with the LR boundaries ($\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05 \text{ m}$), the energy varies over time as a result of the frequency-dependent impedance imposed at the boundaries.

3.2 | ROM Verification

In this section, the ROM is validated against FOM results for both short- and long-duration simulations to highlight the instabilities that typically emerge in long-duration cases. For the $p - \mathbf{v}$ formulation, separate snapshot matrices were constructed from the temporal solutions of acoustic pressure and velocity fields, with POD applied to each independently. In contrast, for the $p - \Phi$ formulation, a single snapshot matrix was created by concatenating the temporal solutions of acoustic pressure and velocity potential column-wise, enabling the use of a unified PSD basis that preserves the coupled structure. In both cases, the full set of time steps from the simulations were retained as snapshots for basis construction, ensuring a consistent temporal resolution in the reduced basis generation. For two-dimensional scenarios, an impulse response length of 50 ms can be classified as short, while durations of 200 to 400 ms can be considered long when compared to typical impulse response lengths observed in physical three-dimensional spaces such as studios and offices, which often range from 200 to 500 ms.

3.2.1 | Short Impulse Response Results

The same representative cases as in the previous section are now used here. To validate ROM against FOM simulations using the formulation $p - \mathbf{v}$ as ground truth, a reduced model size of $N_{\text{rb}} = 122$ was selected. The results, presented in

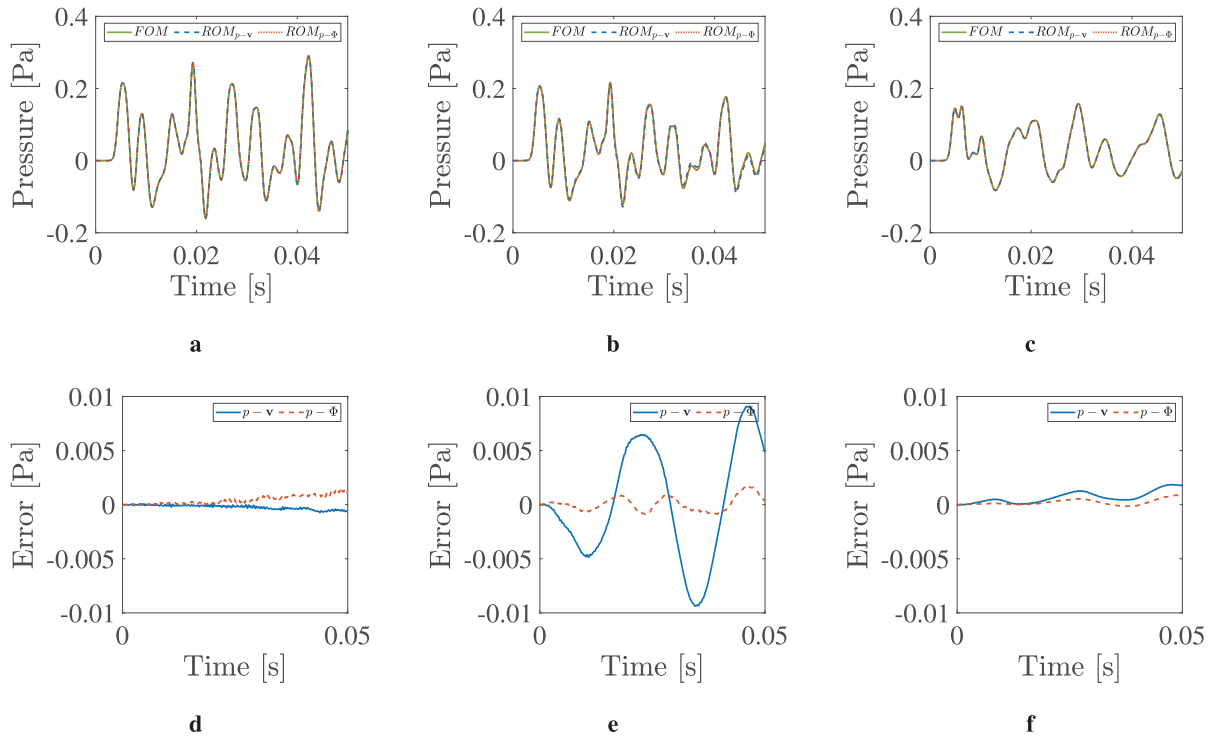


FIGURE 5 | FOM-ROM comparison for impulse response at receiver location $\mathbf{r}_0 = (0.25, 0.22)$ m for $N_{rb} = 122$. (a) PR boundaries, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{mat} = 10,000 \text{ N-s/m}^4$, $d_{mat} = 0.05$ m. Error: (d) PR boundaries, (e) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (f) LR boundaries with $\sigma_{mat} = 10,000 \text{ N-s/m}^4$, $d_{mat} = 0.05$ m.

Figure 5, indicate a good agreement between the impulse response at the receiver point, $\mathbf{r}_0 = (0.25, 0.22)$ m by FOM and ROM in all the cases considered. The error $e(t) = IR(t)_{FOM} - IR(t)_{ROM}$ between two solutions varies over time. To better understand the behavior of the ROM solutions, a numerical experiment was conducted, generating ROMs with the first 500 modes by iteratively varying the ROM size. This approach helps to explore the relationship between the ROM size and its accuracy, providing a comprehensive understanding of the trade-offs involved in reduced-order modeling.

Figure 6 illustrates the error between the FOM and ROM solutions in the L_∞ norm, defined as $\epsilon = \|(P_{FOM} - P_{ROM})\|_\infty$, for two different formulations. The results show that the ROMs constructed using the $p - \Phi$ formulation exhibit a consistent reduction in error as the size of the ROM increases, indicating a clear convergence toward the FOM solution. In contrast, ROMs based on the $p - \mathbf{v}$ formulation do not maintain a strictly monotonic decrease in error with increasing ROM size. This instability is attributed to the loss of structure inherent in the $p - \mathbf{v}$ formulation, which affects the accuracy and stability of the reduced order model.

3.2.2 | Long Impulse Response Results

The test cases from the previous section were extended to simulations of a time span of 0.2 s to validate the performance of the ROM against FOM simulations, using the $p - \mathbf{v}$ formulation as the ground truth. Different ROM sizes were also chosen to investigate precision and stability: (a) PR boundaries with $N_{rb} = 122$ for both formulations, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, where $N_{rb} = 122$ was used for formulation $p - \Phi$ and $N_{rb} = 288$ for formulation $p - \mathbf{v}$, and (c) LR boundaries with $\sigma_{mat} = 10,000 \text{ N-s/m}^4$ and $d_{mat} = 0.05$ m, where $N_{rb} = 200$ was selected for formulation $p - \Phi$ and $N_{rb} = 370$ for the formulation $p - \mathbf{v}$. The results in Figure 7 reveal the presence of unstable ROMs, as indicated by significant errors in the solutions.

Figure 8 shows the error between the FOM and the ROMs in the L_∞ norm defined as $\epsilon = \|(P_{FOM} - P_{ROM})\|_\infty$ using two formulations. For PR boundaries, the results reveal that numerous unstable ROM solutions arise with the $p - \mathbf{v}$ formulation. In contrast, the $p - \Phi$ formulation yields stable ROMs under the same conditions. Our numerical experiments revealed that any error value greater than -2 on the logarithmic scale is indicative of the notable numerical artifacts in the ROM solution.

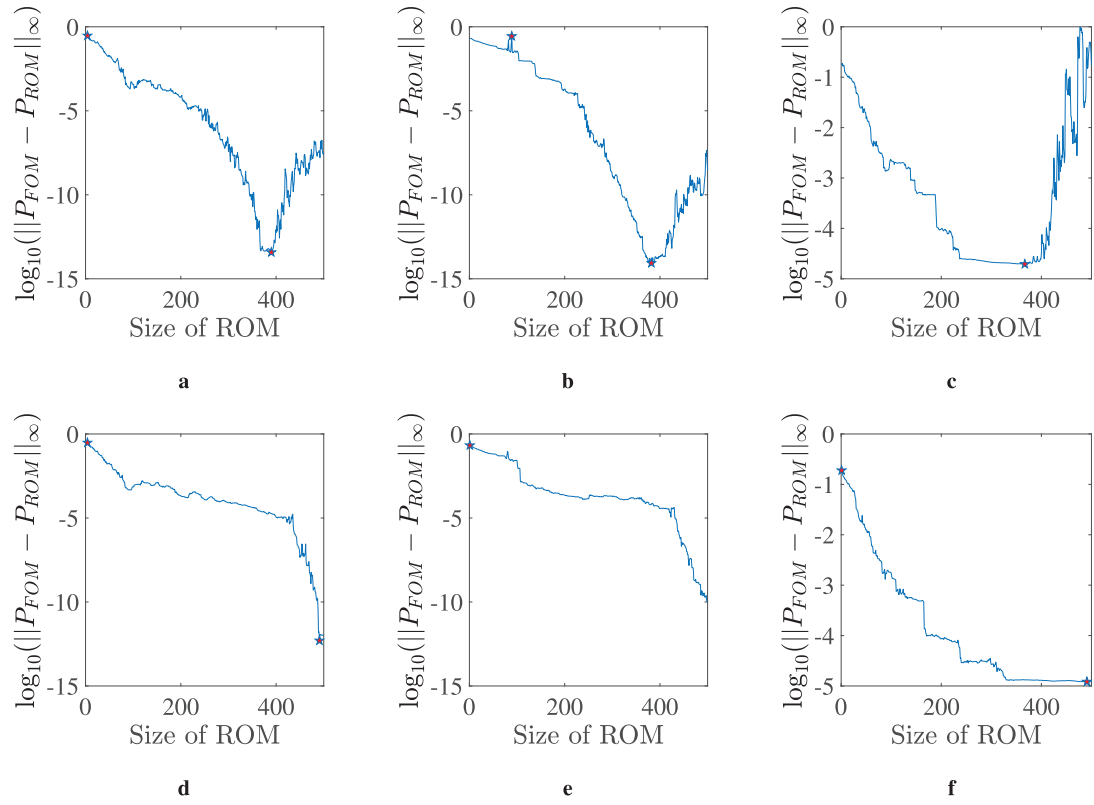


FIGURE 6 | ROMs Error comparison by two formulations ($T = 0.05$ s). $p - \mathbf{v}$ formulation: (a) PR boundaries, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05 \text{ m}$. $p - \Phi$ formulation: (d) PR boundaries, (e) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (f) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05 \text{ m}$.

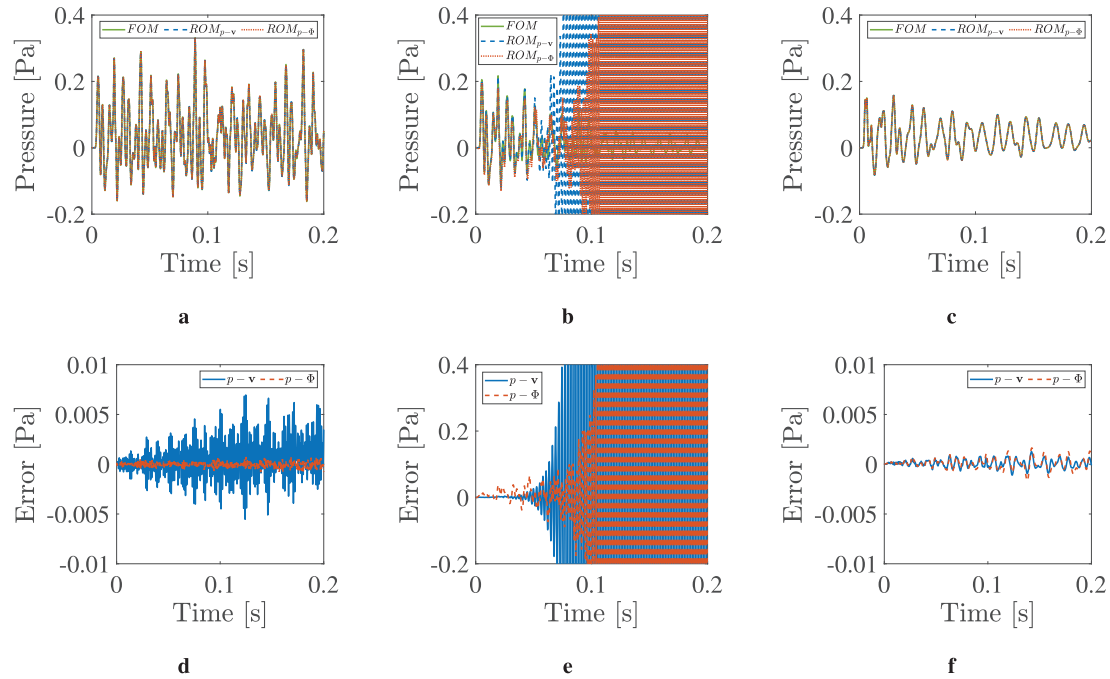


FIGURE 7 | FOM-ROM comparison for impulse response at receiver location $r_0 = (0.25, 0.22)m$. (a) PR boundaries with $N_{\text{rb}} = 122$ for two formulations, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, using $N_{\text{rb}} = 122$ for $p - \Phi$ formulation and $N_{\text{rb}} = 288$ for $p - \mathbf{v}$ formulation. (c) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05m$ using $N_{\text{rb}} = 200$ for $p - \Phi$ formulation and $N_{\text{rb}} = 370$ for $p - \mathbf{v}$ formulation. Error: (d) PR boundaries with $N_{\text{rb}} = 122$ for two formulations, (e) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, using $N_{\text{rb}} = 122$ for $p - \Phi$ formulation and $N_{\text{rb}} = 288$ for $p - \mathbf{v}$ formulation (f) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05m$ using $N_{\text{rb}} = 200$ for $p - \Phi$ formulation and $N_{\text{rb}} = 370$ for $p - \mathbf{v}$ formulation.

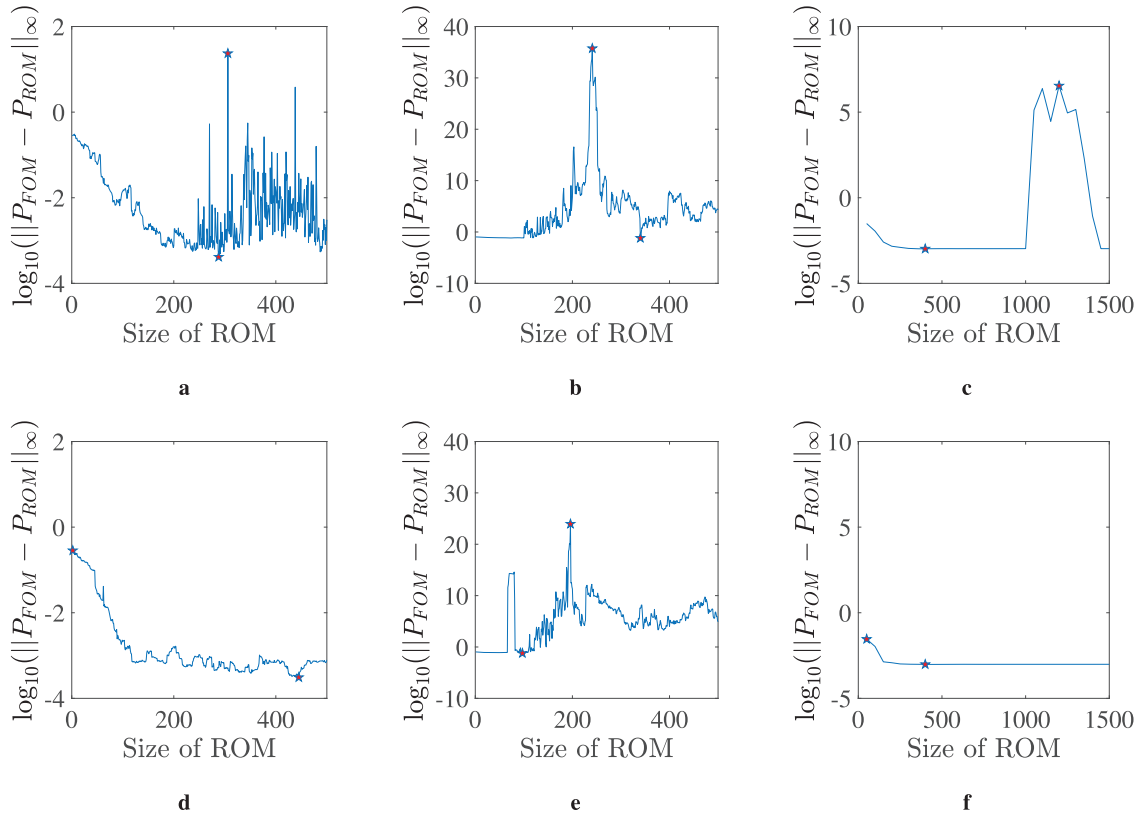


FIGURE 8 | ROMs Error comparison by two formulations ($T = 0.2$ s). $p - \mathbf{v}$ formulation: (a) PR boundaries, (b) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (c) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05$ m. $p - \Phi$ formulation: (d) PR boundaries, (e) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$, and (f) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05$ m.

For FI boundaries, ROMs constructed using both formulations exhibit substantial instabilities, even in the lower spectrum of the solutions. This instability stems from the lack of boundary energy contributions in both formulations, as explained in Section 2.5. In contrast, with LR boundaries, unstable ROMs are not observed in the lower spectrum, but there are some instabilities on the higher side of the spectrum when the $p - \mathbf{v}$ formulation is used. To highlight unstable ROMs on the higher side of the spectrum of solutions, ROM solutions are generated iteratively at intervals of $dN_{\text{rb}} = 50$ across the entire range of ROM sizes. This stability in the lower spectrum of ROM solutions is attributed to the inclusion of boundary energy contributions in the additional formulation designed for LR boundary treatment. However, a limiting level of accuracy with stability persists, regardless of the ROM size when the $p - \Phi$ formulation is used.

3.3 | Stability Analysis

As discussed in Section 2.5, the eigenvalues of the assembled operator matrices for both the FOM and ROM must reside in the negative half of the complex plane to guarantee stability. For the case with PR boundaries, the poles (eigenvalues) of the assembled operator matrices for the FOM and ROM of various sizes are compared for both formulations, as illustrated in Figure 9. Eigenvalues along the positive real axis implies a temporal growth of instabilities. These results align well with the error trends observed in Figure 8.

For damped systems with boundary conditions FI and LR using the formulation $p - \Phi$, the scatter or distribution of eigenvalues between the full-order model (FOM) and ROM appears similar. However, proximity alone might not guarantee stability.

The alignment of the phase angle of the eigenvalues is crucial for systems where damping plays a significant role. Deviations in the unwrapped phase angles between the ROM and FOM shown in Figure 10 indicate potential mismatches in the capture of damping dynamics, which can lead to instability in time-domain simulations. This approach is particularly relevant for systems with complex eigenvalues, as the phase angle reflects the oscillatory behavior of the modes.

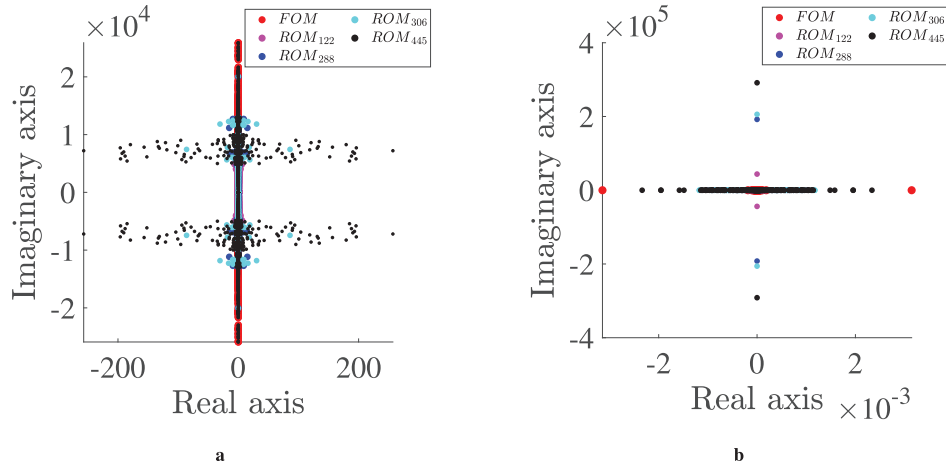


FIGURE 9 | Computed eigenvalues of assembled operator matrices of FOM and ROMs of the system with PR boundaries. (a) $p - \mathbf{v}$ formulation, (b) $p - \Phi$ formulation.

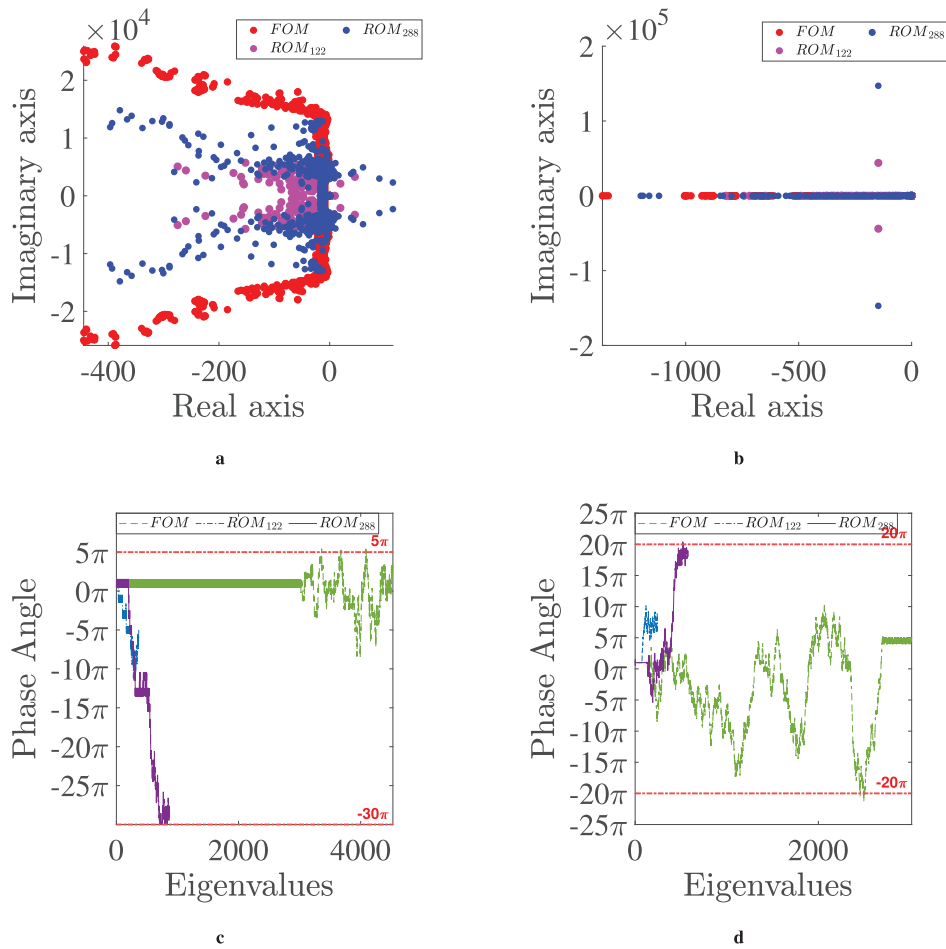


FIGURE 10 | Eigenvalues of assembled operator matrices of FOM and ROMs of the system with FI boundaries. (a) $p - \mathbf{v}$ formulation (b) $p - \Phi$ formulation. Unwrapped Phase angles of eigen values of assembled operator matrices of FOM and ROMs of the system with FI boundaries. (c) $p - \mathbf{v}$ formulation (d) $p - \Phi$ formulation.

3.4 | Stabilized Long Run Results

As detailed in Section 2.6, incorporating boundary energy into solution snapshot matrices for damped systems with FI and LR boundaries, using the modified formulation $p - \Phi$, ensures the construction of stable ROMs. Figure 11 demonstrates the stability achieved for both FI and LR boundary conditions with the modified formulation. Figure 12 shows the comparison of the impulse response at the receiver location $\mathbf{r}_0 = (0.25, 0.22)$ m by stabilized ROMs of various sizes with FOM followed by the associated error. This is further substantiated by Figure 13, which compares the contributions of boundary and interior energy in the damped system. The results emphasize the critical role of the boundary energy in enhancing the stability of the ROM.

A supplementary case of a rectangular domain of $2 \text{ m} \times 2 \text{ m}$ with the boundary condition LR ($\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05 \text{ m}$) is used to further demonstrate the stability achieved by the modified $p - \Phi$ formulation. The model is initiated with a Gaussian pulse pressure distribution (Equation (2)) with $\sigma = 0.3 \text{ m}^2$. This yields a broadband initial pulse that has a -10 dB upper cutoff frequency of around 700 Hz . The pulse is centered at $\mathbf{x}_0 = (0.7, 1.2) \text{ m}$. The domain is meshed with an unstructured mesh of triangular elements and $P = 4$ basis functions are used. The resolution of the mesh is chosen so that there are approximately 7 points per wavelength (PPW) at 700 Hz . This yields a mesh that has 88 elements and $N_{\text{rb}} = 761$ DOFs in total. The simulation time is 200 ms . The time step $\delta t = 1.5 \times 10^{-5} \text{ s}$ was chosen according to the Courant-Friedrichs-Lewy (CFL) condition of 0.2. ROM solutions are constructed at intervals of $dN_{\text{rb}} = 5$ starting from $N_{\text{rb}} = 50$ across the entire range of ROM sizes using different formulations and are compared in Figure 14. ROMs constructed by modified $p - \Phi$ formulation are stable in the full spectrum of ROM solutions.

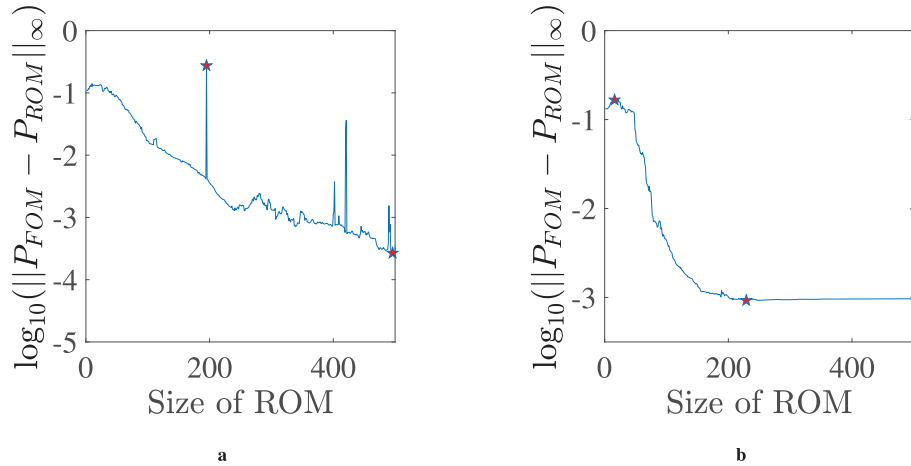


FIGURE 11 | ROMs Error by modified $p - \Phi$ formulation ($T = 0.2 \text{ s}$). (a) FI boundaries with $Z = 16,000 \text{ N-s/m}^4$. (b) LR boundaries with $\sigma_{\text{mat}} = 10,000 \text{ N-s/m}^4$, $d_{\text{mat}} = 0.05 \text{ m}$.

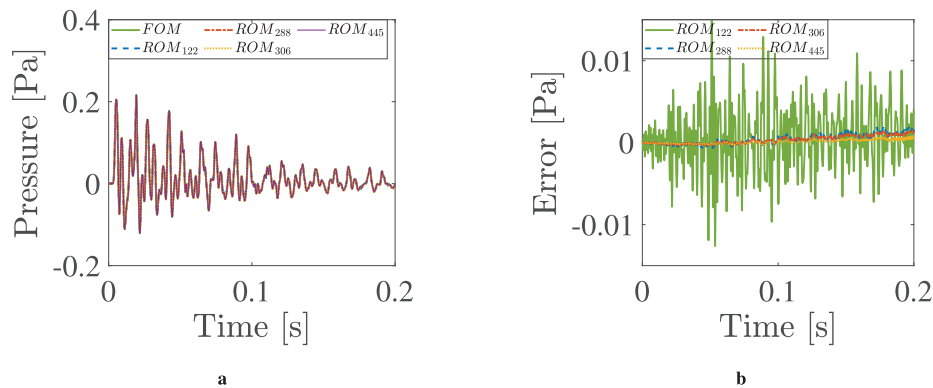


FIGURE 12 | (a) FOM-ROM comparison for impulse response at receiver location $\mathbf{r}_0 = (0.25, 0.22) \text{ m}$ for FI boundaries with $Z = 16,000 \text{ N-s/m}^4$. (b) Associated Error.

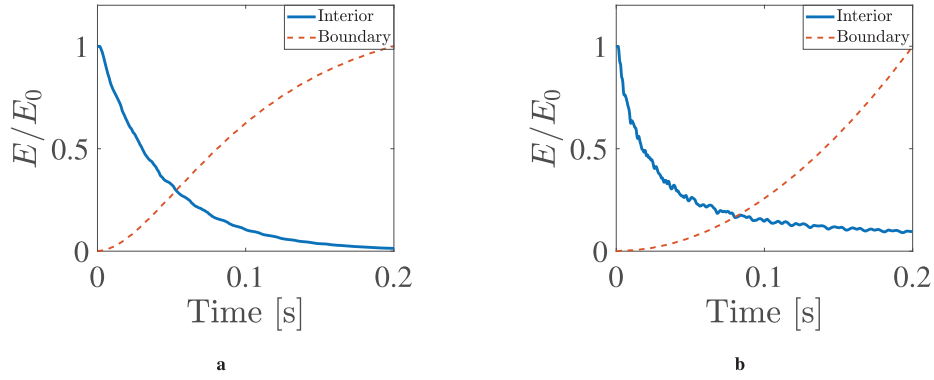


FIGURE 13 | Energy conservation ($T = 0.2$ s). (a) FI boundaries with $Z = 16,000\text{N-s/m}^4$, (b) LR boundaries with $\sigma_{\text{mat}} = 10,000\text{N-s/m}^4$, $d_{\text{mat}} = 0.05$ m.

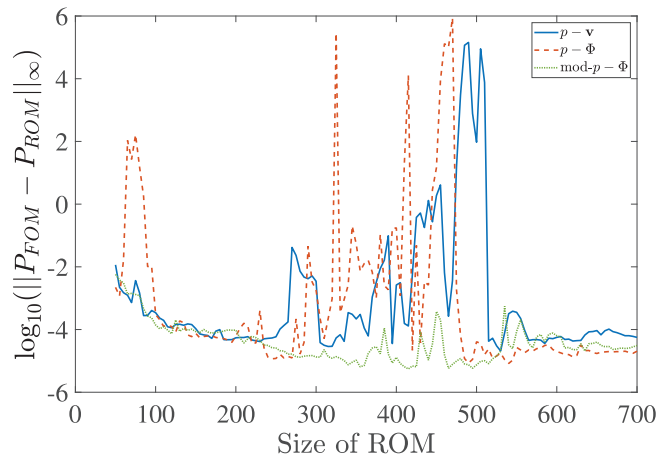


FIGURE 14 | Comparison of ROMs ($dN_{\text{rb}} = 5$) constructed by three formulations: LR boundaries with $\sigma_{\text{mat}} = 10,000\text{N-s/m}^4$, $d_{\text{mat}} = 0.05$ m. Simulation parameters are: source location $\mathbf{x}_0 = (0.7, 1.2)$ m, $P = 4$, $\delta t = 1.5 \times 10^{-5}$ s, $PPW = 7$, $N_e = 88$, $N_{\text{fb}} = 761$ DOFs.

3.5 | Speedup of ROM

The primary objective of using ROM is to achieve a significant increase in computational speed compared to FOM. Figure 15 illustrates the CPU times required for ROM calculations in three formulations as a function of the ROM size. The corresponding error, ϵ , is also plotted on a logarithmic scale, providing a clear visualization of the trade-off between computational efficiency and accuracy.

$$p - \Phi p - \mathbf{v}.$$

4 | Discussion

In this study, a faster, more efficient, and stable method for constructing reduced-order models (ROMs) is demonstrated using the structure-preserving $p - \Phi$ formulation. This approach directly addresses stability issues in long-duration time-domain simulations, offering a systematic way to improve robustness while maintaining computational efficiency. As shown in Figure 8, the $p - \Phi$ formulation improves ROM stability under pressure-release (PR) boundary conditions, with reduced errors and better agreement with full-order model (FOM) results. Figures 11 and 14 confirm similar improvements for frequency-independent (FI) and locally reactive (LR) boundary conditions. This is primarily due to the inclusion of boundary energy in the snapshot matrices, which helps the reduced basis capture the damping effects introduced by the boundaries.

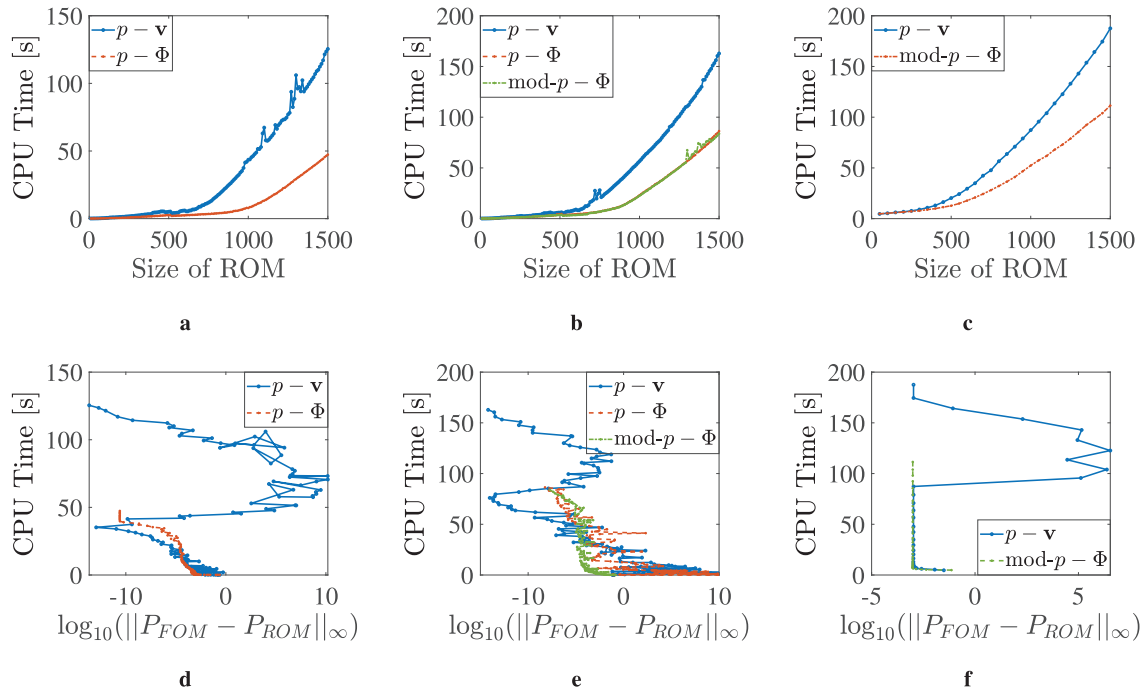


FIGURE 15 | Comparison of CPU time as function of size of ROMs ($dN_{rb} = 10$) by two formulations ($T = 0.2$ s). (a) PR boundaries, (b) FI boundaries with $Z = 16,000\text{N-s/m}^4$, and (c) LR boundaries with $\sigma_{mat} = 10,000\text{N-s/m}^4$, $d_{mat} = 0.05$ m ($dN_{rb} = 50$). Comparison of CPU time as function of error of ROMs by two formulations (d) PR boundaries, (e) FI boundaries with $Z = 16,000\text{N-s/m}^4$, and (f) LR boundaries with $\sigma_{mat} = 10,000\text{N-s/m}^4$, $d_{mat} = 0.05$ m ($dN_{rb} = 50$).

Although the structure-preserving approach leads to stable ROMs across a wide range of sizes, one specific case stands out. Both the non-stabilized and stabilized ROMs show the highest error at ROM size 195. In the non-stabilized case, the error increases steadily and peaks at 195, while in the stabilized ROM, the error stays low for almost all sizes except for a distinct spike again at 195. Since this behavior appears in both cases, it is likely not related to instability in the reduced system, but rather to how the basis is constructed.

At this truncation point, the reduced basis may fail to represent a dynamically important mode whose interaction with the retained modes is necessary to capture the full wave dynamics. This behavior is consistent with what has been described in ROM literature as mode truncation sensitivity, where certain truncation indices produce temporarily ill-conditioned reduced spaces and result in increased error. This type of non-monotonic error trend with increasing POD basis size has been studied in the context of POD basis stability, where changes in subspace geometry can lead to temporary increases in error norm for certain truncation indices [80].

While boundary energy dissipation remains important for stability, the aligned error spike in both ROM versions points to limitations in the reduced basis itself. In the standard $p - v$ formulation, unstable ROMs are more common at larger sizes, but they can generally be avoided by selecting modes from the lower end. However, these findings show that issues with modal representation can also occur at intermediate ROM sizes. This highlights the need to ensure that the reduced basis is not just stable, but also capable of capturing all key dynamics, even at specific truncation choices, when constructing ROMs for time-domain acoustic simulations.

The desired attributes of ROM by the proposed formulation, such as speedup and accuracy, as shown in Figure 15, are consistent with previous work [44, 45].

5 | Conclusions

This study presents a novel methodology for constructing ROMs by incorporating boundary-energy contributions, providing valuable insights into the stability of reduced-order systems with complex boundary conditions. The proposed approach effectively maintains stability while preserving the structure of the time-domain model. Future work could

explore alternative ROM construction strategies to ensure stability across a broader range of damping configurations and extend their applicability to three-dimensional systems.

The key contributions of this work are fourfold:

- The proposed formulation not only enhances the efficiency of FOM computations but also significantly accelerates the overall MOR workflow.
- It is compatible with a wide range of boundary conditions, including highly reflective, frequency independent, and frequency-dependent cases.
- The formulation ensures stable ROMs with minimal computational overhead, enabling real-time applications in virtual acoustics.
- The study highlights the often overlooked challenge of selecting an optimal reduced basis size, balancing computational speedup, accuracy, reduction potential, and ROM stability in the time domain.

By addressing stability challenges through the proposal of a new formulation and the demonstration of an efficient structure-preserving approach, this study contributes to the advancement of model order reduction techniques for time-domain room acoustics simulations.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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